Amoebas of maximal area.

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Abstract

To any algebraic curve A in $(\mathbb{C}^*)^2$ one may associate a closed infinite region $\mathcal{A}$ in $\mathbb{R}^2$ called *the amoeba* of A. The amoebas of different curves of the same degree come in different shapes and sizes. All amoebas in $(\mathbb{R}^*)^2$ have finite area and, furthermore, there is an upper bound on the area in terms of the degree of the curve.

The subject of this paper is the curves in $(\mathbb{C}^*)^2$ whose amoebas are of the maximal area. We show that up to multiplication by a constant in $(\mathbb{C}^*)^2$ such curves are defined over $\mathbb{R}$ and, furthermore, that their real loci are isotopic to so-called *Harnack curves*.

1 Introduction.

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial, $f(z_1, z_2) = \sum\limits_{j,k} a_{jk} z_1^j z_2^k$. Its zero set in $(\mathbb{C}^*)^2$ is a curve $A = f^{-1}(0) \cap (\mathbb{C}^*)^2$ (where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$). Let $\Delta \subset \mathbb{R}^2$ be the Newton polygon of f, i.e. the convex hull of $\{(j,k) \mid a_{jk} \neq 0\}$. Gelfand, Kapranov and Zelevinski introduced one more object associated to f.

Definition 1 (Gelfand, Kapranov, Zelevinski [3]). [1] The amoeba $\mathcal{A} \subset \mathbb{R}^2$ of f is $\mathrm{Log}(A)$, where $\mathrm{Log} : (\mathbb{C}^*)^2 \to \mathbb{R}^2$, $(z_1, z_2) \mapsto (\log|z_1|, \log|z_2|)$.

It was remarked in [3] that every component of $\mathbb{R}^2 \setminus \mathcal{A}$ is open and convex in $\mathbb{R}^2$. In particular, $\mathcal{A}$ is closed and its (Lebesgue) area is well-defined.

Note that $\mathcal{A}$ is never bounded in $\mathbb{R}^2$, since $f^{-1}(0)$ must intersect the coordinate axes in $\mathbb{C}^2$. However it was shown by Passare and Rullgård [8] that the area of $\mathcal{A}$ is always finite. Furthermore, it is bounded in terms of Δ.

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[1] In this paper we restrict our attention to functions of two variables. Amoebas are defined for functions of any number of variables.

Theorem (Passare, Rullgård [8]).

$$\operatorname{Area}(\mathcal{A}) \leq \pi^2 \operatorname{Area}(\Delta). \tag{1}$$

The main result of this paper is the extremal property of this inequality.

We say that a curve A is defined over $\mathbb{R}$ if it is invariant under the complex conjugation $\operatorname{conj} : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$, $(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$. In this case we may consider the real part of the curve $\mathbb{R}A = A \cap (\mathbb{R}^*)^2$ which is a real algebraic curve. We say that a curve A is real up to multiplication by a constant if there exist constants $b_1, b_2 \in \mathbb{C}^*$ such that $(b_1, b_2) \times A \subset (\mathbb{C}^*)^2$ is defined over $\mathbb{R}$. The condition that A is real up to multiplication by a constant is equivalent to the condition that there exist $a, b_1, b_2 \in \mathbb{C}^*$ such that the polynomial $af(\frac{z_1}{b_1}, \frac{z_2}{b_2})$ has real coefficients. In this case we may also consider the real part $\mathbb{R}A = \{(x_1, x_2) \in (\mathbb{R}^*)^2 \mid af(\frac{x_1}{b_1}, \frac{x_2}{b_2}) = 0\}$. We say that a map is at most 2-1 if the inverse image of any point in the target consists of at most 2 points. The main result of this paper is the following theorem.

Theorem 1. *Suppose that* $\operatorname{Area}(\Delta) > 0$. *Then the following conditions are equivalent.*

1. $\operatorname{Area}(\mathcal{A}) = \pi^2 \operatorname{Area}(\Delta)$.
2. *The map* $\operatorname{Log}|_A : A \to \mathbb{R}^2$ *is at most 2-1 and* A *is real up to multiplication by a constant.*
3. *The curve* A *is real up to multiplication by a constant and its real part* $\mathbb{R}A$ *is a (possibly singular) Harnack curve (see Definitions 2 and 3) for the Newton polygon* Δ.

Furthermore, these conditions imply that the non-singular locus of $\mathbb{R}A$ *coincides with* $A \cap \operatorname{Log}^{-1}(\partial\mathcal{A})$.

Corollary 1. *The inequality* (1) *is sharp for any Newton polygon* Δ.

The corollary follows from Theorem 1 and the Harnack-Itenberg-Viro Theorem (see section 2) on existence of Harnack curves.

Remark 1. A curve that is real up to multiplication by a constant may have more than one real part (other real parts may come as a result of multiplication by different constants). For instance, if f is a real polynomial which contains only even powers of z_2 then the pullback of f under $(z_1, z_2) \mapsto (z_1, iz_2)$ is a real polynomial with a different real part.

The theorem implies that a Harnack curve is real up to multiplication by a constant in a unique way. Indeed, the choice of the real part is determined by the identity $\mathbb{R}A = A \cap \operatorname{Log}^{-1}(\partial\mathcal{A})$.

2 Harnack curves in $(\mathbb{R}^*)^2$.

Let us fix a convex polygon $\Delta \subset \mathbb{R}^2$ whose vertices have integer coordinates. Consider all possible real polynomials f whose Newton polygon is Δ. The same polynomial f may be viewed both as a function $(\mathbb{C}^*)^2 \to \mathbb{C}$ and as a function $(\mathbb{R}^*)^2 \to \mathbb{R}$.

Let $\mathbb{R}A$ be the zero set of f in $(\mathbb{R}^*)^2$. Equivalently, $\mathbb{R}A$ is a real part of the zero set A of f in $(\mathbb{C}^*)^2$. For a generic choice of coefficients of f the curve $\mathbb{R}A$ is smooth. However the topology of $((\mathbb{R}^*)^2, \mathbb{R}A)$ is different for different choices of coefficients of f. In particular, the number of components of $\mathbb{R}A$ may be different. Also the mutual position of the components may be different.

We may compactify the above setup. Recall (see e.g. [3]) that the polygon Δ determines a toric surface $\mathbb{C}T_\Delta \supset (\mathbb{C}^*)^2$. We denote the real part of $\mathbb{C}T_\Delta$ with $\mathbb{R}T_\Delta \supset (\mathbb{R}^*)^2$. The surface $\mathbb{C}T_\Delta$ is a compactification of $(\mathbb{C}^*)^2$. Furthermore, the complement $\mathbb{C}T_\Delta \smallsetminus (\mathbb{C}^*)^2$ is a union of n (non-disjoint) lines, where n is the number of sides of Δ. Similarly, $\mathbb{R}T_\Delta \smallsetminus (\mathbb{R}^*)^2$ is a union of n real lines $l_1, \dots, l_n$. These lines are called the *axes* of $\mathbb{R}T_\Delta$. We assume that the indexing of l_k is consistent with the natural cyclic order on the sides of Δ.

The closure $\bar{A}$ of $A \subset (\mathbb{C}^*)^2 \subset \mathbb{C}T_\Delta$ in $\mathbb{C}T_\Delta$ is a compact curve whose real part is $\mathbb{R}\bar{A} \supset \mathbb{R}A$. The topology of the triad $(\mathbb{R}T_\Delta; \mathbb{R}\bar{A}, l_1, \cup \dots \cup l_n)$ carries all topological information on arrangement of $\mathbb{R}A$ in $(\mathbb{R}^*)^2$.

The upper bound on the number of components of $\mathbb{R}\bar{A} \subset \mathbb{R}T_\Delta$ is provided by Harnack's inequality [4]. This number is never greater than one plus the genus of A. Recall that by [6] the genus of A is equal to the number of lattice points in the interior of Δ. We denote this number with g.

To deduce the upper bound on the number of components of $\mathbb{R}A \subset (\mathbb{R}^*)^2$ we recall that $\mathbb{R}A = \mathbb{R}\bar{A} \smallsetminus (l_1 \cup \dots \cup l_n)$, where l_k corresponds to a side δ_k of Δ. Let d_k be the integer length of δ_k, i.e. the number of lattice points inside δ_k plus one. Note that this length is an $SL(2, \mathbb{Z})$-invariant. The curve $\mathbb{R}\bar{A}$ and the axis l_k intersect in no more that d_k points, since d_k is the intersection number of their complexifications. Therefore, $\mathbb{R}A$ has no more than $g + \sum\limits_{k=1}^{n} d_k$ components.

Definition 2 (Harnack curves, cf. [7]). A non-singular curve $\mathbb{R}A \subset (\mathbb{R}^*)^2$ with the Newton polygon Δ is called a *Harnack curve* if all the following conditions hold.

- The number of components of $\mathbb{R}\bar{A}$ is equal to $g+1$ (where g is the number of lattice points in the interior of Δ).
- All components of $\mathbb{R}\bar{A}$ but one do not intersect $l_1 \cup \dots \cup l_n$.
- A component C of $\mathbb{R}\bar{A}$ can be divided into n consecutive (with respect to the cyclic order on C) arcs $\alpha_1, \dots, \alpha_n$ so that for each k the intersections $\alpha_k \cap l_k$ consists of d_k points, while $\alpha_k \cap l_j = \emptyset$, $j \neq k$.

Note that the first two conditions imply that the number of components of a Harnack curve $\mathbb{R}A$ is equal to $g + \sum\limits_{k=1}^{n} d_k$.

Theorem (Mikhalkin [7]). *For each Newton polygon Δ the topological type of the triad $(\mathbb{R}T_\Delta; \mathbb{R}\bar{A}, l_1 \cup \cdots \cup l_n)$ is unique if $\mathbb{R}A$ is a Harnack curve.*

Note that the above theorem implies that the topological type of the pair $((\mathbb{R}^*)^2, \mathbb{R}A)$ is also unique for each Δ.

Theorem (Harnack, Itenberg, Viro, [4], [5], [7]). *Harnack curves exist for any Newton polygon Δ.*

Harnack [4] proved this theorem for plane projective curves of arbitrary degree d. In our language this corresponds to the case when Δ is a triangle whose vertices are $(0,0)$, $(d,0)$, $(0,d)$. Harnack's example was generalized to arbitrary Newton polyhedra Δ with the help of Viro's patchworking described in [5], see Corollary A4 in [7]. The Harnack curves are a special case of the so-called T-curves, see [5].

We refer to [5] and [7] for illustrations of Harnack curves.

Recall that a point $p \in \mathbb{R}A \subset (\mathbb{R}^*)^2$ is called an ordinary real isolated double point of $\mathbb{R}A$ (or an A_1^+-point, see [2]) if there exist local coordinates x_1, x_2 at $p \subset (\mathbb{R}^*)^2$ such that A is locally defined by equation $x_1^2 + x_2^2 = 0$.

Definition 3 (Singular Harnack curves). A singular curve $\mathbb{R}A \subset (\mathbb{R}^*)^2$ with the Newton polygon Δ is called a singular *Harnack curve* if

- the only singular points of $\mathbb{R}A$ are A_1^+-points (ordinary real isolated double points);
- the result of replacing of the singular points of $\mathbb{R}A$ with small ovals (which corresponds to replacing with the locus $x_1^2 + x_2^2 = \epsilon, \epsilon > 0$ in the local coordinates) gives a Harnack curve for Δ.

In other words, a singular Harnack curve is the result of contraction to points of some ovals of a non-singular Harnack curve.

3 Monge-Ampère measure on $\mathcal{A}$.

In the next section we prove the equivalence of conditions 1 and 2 in the main theorem. The proof is an extension of the proof of the inequality (1) given in [8]. We recapture in this section the main points in this proof. The idea is to construct a measure on the amoeba $\mathcal{A}$, whose total mass is related to Δ and which can be computed explicitly in terms of the hypersurface A. This measure will be obtained as the real Monge-Ampère measure of a certain convex function associated to f.

We indicate briefly the definition of the real Monge-Ampère operator. Details may be found in [9]. Suppose u is a smooth convex function defined in $\mathbb{R}^n$. Then $\operatorname{grad} u$ defines a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. The Monge-Ampère measure $\mathrm{M}\, u$ of u is defined by $\mathrm{M}\, u(E) = \lambda(\operatorname{grad} u(E))$ for any Borel set E, where λ denotes Lebesgue measure on $\mathbb{R}^n$. That this is actually a measure requires a proof, since $\operatorname{grad} u$ is in general not 1-to-1. If u is convex but not necessarily smooth,

grad u can still be defined as a multifunction, and the Monge-Ampère measure of u is defined as in the smooth case. For smooth functions the Monge-Ampère measure is given by the determinant of the Hessian matrix,

$$\mu = |\operatorname{Hess}(u)|\lambda,$$

where λ is the Lebesgue measure.

Suppose now that f is a given polynomial in two variables and define

$$N_f(x) = \frac{1}{(2\pi i)^2}\int_{\operatorname{Log}^{-1}(x)} \frac{\log|f(z)|\,dz_1\,dz_2}{z_1 z_2}.$$

This is a real-valued function defined in $\mathbb{R}^2$, which is convex because $\log|f(z)|$ is plurisubharmonic. Define μ to be the Monge-Ampère measure of N_f.

Lemma 1. *The measure μ has its support in $\mathcal{A}$ and its total mass is equal to the area of Δ.*

Proof. It is not difficult to show that N_f is affine linear in each connected component of $\mathbb{R}^2 \setminus \mathcal{A}$ and that the gradient image grad $N_f(\mathbb{R}^2)$ is equal to Δ minus some of its boundary points. This readily implies the statement. For details we refer to [8]. □

Let F denote the set of critical values of the mapping $\operatorname{Log} : A \to \mathbb{R}^2$. Pick a point $x_0 \in \mathcal{A} \setminus F$ and functions ϕ_j, ψ_j defined in a neighborhood V of x_0, where j ranges from 1 to n and n is the cardinality of $\operatorname{Log}^{-1}(x_0) \cap A$, such that $A \cap \operatorname{Log}^{-1}(V) = \cup_{j=1}^{n}\{(\exp(x_1 + i\phi_j(x)), \exp(x_2 + i\psi_j(x))); x = (x_1, x_2) \in V\}$. The main step in the proof of the inequality is the following computation.

Lemma 2. *With notations as above we have*

$$\operatorname{Hess}(N_f) = \frac{1}{2\pi}\sum_{j=1}^{n} \pm \begin{pmatrix} \partial\psi_j/\partial x_1 & \partial\psi_j/\partial x_2 \\ -\partial\phi_j/\partial x_1 & -\partial\phi_j/\partial x_2 \end{pmatrix}. \qquad (2)$$

The signs depend on the signs of the intersection numbers between $\operatorname{Log}^{-1}(x_0)$ and A. Each term in the sum is a symmetric, positive definite matrix with determinant equal to 1.

For the proof we refer to [8]. We remark that the fact that the matrices are symmetric with determinant equal to 1 follows immediately when we know that A is a complex analytic curve. The two last lemmas immediately imply the inequality (1) via the following corollary.

Corollary 2. *If λ denotes Lebesgue measure in $\mathbb{R}^2$, then $\mu \geq (\lambda/\pi^2)|_{\mathcal{A}}$. Hence the area of $\mathcal{A}$ is not greater than π^2 times the area of Δ.*

Proof. It is not difficult to show that for 2×2 symmetric, positive definite matrices M_1, M_2 the inequality

$$\sqrt{\det(M_1 + M_2)} \geq \sqrt{\det M_1} + \sqrt{\det M_2} \tag{3}$$

holds, with equality precisely if M_1 and M_2 are real multiples of each other. Applying this to the sum (2) and using the fact that it contains at least two terms for all $x_0 \in \mathcal{A} \setminus F$, the first statement follows. Combining this with Lemma 1 yields the second part. $\square$

Remark 2. The inequality used in the previous proof follows as a special case of an inequality for positive definite matrices of arbitrary size, analogous to the Alexandrov-Fenchel inequality for mixed volumes. The general inequality can be found in [1].

4 Proof of Theorem 1: conditions 1 and 2 are equivalent.

We are now ready to prove the equivalence of conditions 1 and 2. Note that by Corollary 2, $\operatorname{Area}(\mathcal{A}) = \pi^2 \operatorname{Area}(\Delta)$ if and only if $\mu = (\lambda/\pi^2)|_{\mathcal{A}}$.

4.1 Implication $1 \implies 2$.

Suppose that $\mu = (\lambda/\pi^2)|_{\mathcal{A}}$. We first show that f is irreducible.

Lemma 3. *If $\mu = (\lambda/\pi^2)|_{\mathcal{A}}$, then f is irreducible.*

Proof. Let K, L be compact convex subsets of $\mathbb{R}^2$. From the monotonicity properties of mixed volumes it follows that $\operatorname{Area}(K + L) \geq \operatorname{Area}(K) + \operatorname{Area}(L)$ with strict inequality holding unless one of K, L is a point or K and L are two parallel segments. Assume now that we have a non-trivial factorization $f = gh$ and let Δ_g, Δ_h denote the Newton polytopes and $\mathcal{A}_g, \mathcal{A}_h$ the amoebas of g and h respectively. From Lemma 1 it follows that $\operatorname{Area}(\mathcal{A}) = \pi^2 \operatorname{Area}(\Delta)$. On the other hand, since $\mathcal{A} = \mathcal{A}_g \cup \mathcal{A}_h$ and $\Delta = \Delta_g + \Delta_h$, it follows from Corollary 2 that

$$\operatorname{Area}(\mathcal{A}) \leq \operatorname{Area}(\mathcal{A}_g) + \operatorname{Area}(\mathcal{A}_h) \leq \pi^2(\operatorname{Area}(\Delta_g) + \operatorname{Area}(\Delta_h)) < \pi^2 \operatorname{Area}(\Delta).$$

This is a contradiction. $\square$

From (3) it follows that for equality to hold in Corollary 2 it is necessary that $\operatorname{Log}^{-1}(x)$ intersects A in at most two points for all $x \notin F$. Hence the sum (2) contains two terms with opposite signs. For equality to hold in (3) applied to the sum (2) it is necessary that $\operatorname{grad} \phi_1 = -\operatorname{grad} \phi_2$ and $\operatorname{grad} \psi_1 = -\operatorname{grad} \psi_2$. After a multiplication of each coordinate by a constant we may assume that $\phi_1 = -\phi_2, \psi_1 = -\psi_2$ in a neighborhood of a given point in $\mathcal{A} \setminus F$. (The existence

of such points is guaranteed by the assumption that Area(Δ) and hence Area($\mathcal{A}$) is positive.) But then $f(z)$ and $\overline{f(\bar{z})}$ have a common factor, and hence coincide up to a multiplicative constant since they are irreducible. Multiplying f by a suitable constant, we obtain a polynomial with real coefficients.

To complete the proof we must show that $\mathrm{Log}^{-1}(x_0)$ intersects A in at most two points for all $x_0 \in F$. Note that $\mathrm{Log}^{-1}(x_0) \cap A$ cannot contain more than 2 isolated points. Indeed, a small neighborhood in A of an isolated point in $\mathrm{Log}^{-1}(x_0) \cap A$ is mapped by Log either onto a neighborhood of x_0, or in a 2-to-1 fashion onto a half-disk with x_0 on its boundary. In any case, the presence of more than 2 isolated points would imply that $\mathrm{Log}^{-1}(x) \cap A$ contains more than two points for some $x \notin F$, which is a contradiction.

If $\mathrm{Log}^{-1}(x_0) \cap f^{-1}(0)$ contains a curve γ we consider two different cases. If γ is of the form $\mathrm{Log}^{-1}(x_0) \cap \{z_1^j z_2^k = c\}$ for some $(j,k) \in \mathbb{Z}^2$ and $c \in \mathbb{C}$, then f contains the factor $z_1^j z_2^k - c$, which is impossible by Lemma 3. Otherwise, $t\gamma := \{(t_1 z_1, t_2 z_2); (z_1, z_2) \in \gamma\}$ intersects γ for all t in an open set in the real torus $\mathbf{T}^2$. By Theorem 5 in [8] (cf. the proof of Lemma 4) this implies that μ has a point mass at x_0, contradicting the assumptions. Hence we have shown that $\mathrm{Log} : A \to \mathbb{R}^2$ is at most 2-to-1.

4.2 Implication $2 \implies 1$.

Conversely, assume that $\mathrm{Log} : A \to \mathbb{R}^2$ is at most 2-to-1 and that f has real coefficients. Since $\mathcal{A}$ and μ are invariant under the changes of variables permitted in the theorem, this is no loss of generality. Then the sum (2) has two terms. Since A is invariant under complex conjugation of the variables, it follows that $\phi_1 = -\phi_2, \psi_1 = -\psi_2$, hence the two terms are actually equal. This shows immediately that $\mu = (\lambda/\pi^2)|_{\mathcal{A}}$ outside F. By the following Lemma neither μ nor λ has any mass on F, so this equality holds everywhere.

Lemma 4. *If $\mathrm{Log}^{-1}(x) \cap A$ is a finite set for all x, then μ has no mass on F.*

Proof. In Theorem 5 in [8] it is shown that $\mu(E)$ is proportional to the average number of solutions in $\mathrm{Log}^{-1}(E)$ to the system of equations

$$f(z_1, z_2) = f(t_1 z_1, t_2 z_2) = 0 \tag{4}$$

as (t_1, t_2) ranges over the real torus $\mathbf{T}^2 = \{t \in \mathbb{C}^2; |t_1| = |t_2| = 1\}$. Note that the set of critical values of the mapping $A \to \mathbb{R}^2 : (z_1, z_2) \mapsto (|z_1|^2, |z_2|^2)$ is a semialgebraic set. Thus it is contained in a real-algebraic curve $\tilde{F}$.

Consider the product space $\mathbb{C}^2 \times \mathbf{T}^2$ with the two projections π_1 and π_2 onto $\mathbb{R}^2$ and $\mathbf{T}^2$ defined by $\pi_1(z,t) = (|z_1|^2, |z_2|^2)$ and $\pi_2(z,t) = t$. Let $C = \pi_1^{-1}(\tilde{F}) \cap \{f(z_1, z_2) = f(t_1 z_1, t_2 z_2) = 0\} \subset \mathbb{C}^2 \times \mathbf{T}^2$. Since the map $\pi_1 : C \to \tilde{F}$ has discrete fibers, it follows that C is a real curve. Hence $\pi_2(C)$ is a null set in $\mathbf{T}^2$. Since the equation (4) has no solutions in $\mathrm{Log}^{-1}(F)$ for t outside $\pi_2(C)$, it follows that $\mu(F) = 0$ as required. $\square$

5 Proof of Theorem 1: conditions 2 and 3 are equivalent.

5.1 Implication $2 \implies 3$.

By our assumption A is real up to multiplication by a constant. Thus multiplying by a suitable constant we may assume that A is already defined over $\mathbb{R}$. In this case we may define the real part $\mathbb{R}A$ as the fixed point set of the involution of complex conjugation $\mathrm{conj} : (z_1, z_2) \mapsto (\bar{z_1}, \bar{z_2})$ restricted to A.

Let $\nu : \tilde{A} \to A$ be the normalization of the curve A. The involution $\mathrm{conj}|_A$ can be lifted to an involution $\mathrm{conj}_{\tilde{A}}$ on the Riemann surface $\tilde{A}$. Let $\mathbb{R}\tilde{A}$ be the real part of $\tilde{A}$. Note that $\nu(\mathbb{R}\tilde{A}) \subset \mathbb{R}A$, but real isolated (singular) points of $\mathbb{R}A$ are not contained in $\nu(\mathbb{R}\tilde{A})$.

Since $\mathrm{Log}|_A$ is at most 2-1 we can view the map $\mathrm{Log} \circ \nu : \tilde{A} \to \mathcal{A}$ as a branched double covering. Let $F \subset \mathcal{A}$ be the branch locus of this covering, i.e. the set of points whose inverse image under $\mathrm{Log}|_A$ consists of one point.

Lemma 5. *The involution* $\mathrm{conj}_{\tilde{A}}$ *is the deck transformation of the branched double covering* $\mathrm{Log} \circ \nu$.

Proof. The Lemma follows from the fact that Log maps conjugate points to the same point, $\mathrm{Log} \circ \mathrm{conj} = \mathrm{Log}$. ☐

Corollary 3. $\mathcal{A} = \tilde{A}/\mathrm{conj}_{\tilde{A}}$, *while* $F = \mathrm{Log}(\nu(\mathbb{R}\tilde{A})) = \partial\mathcal{A}$.

Proof. The curve $\tilde{A}$ is non-singular and therefore $\tilde{A}/\mathrm{conj}_{\tilde{A}}$ is a smooth surface with the boundary $\mathbb{R}\tilde{A}$. ☐

Thus $\partial\mathcal{A}$ consists of the images of components of $\mathbb{R}\tilde{A}$. These components are of two types, closed components, called *ovals*, and non-compact components. Accordingly, each oval of $\mathbb{R}A$ which does not contain singular points corresponds to a hole in $\mathcal{A}$.

Consider first the case when A is a non-singular curve, so that $\tilde{A} = A$. Let l be the number of ovals of $\mathbb{R}A$. Then $\chi(\mathcal{A}) = 1 - l$, where χ stands for the homology Euler characteristic, i.e. the alternated sum of Betti numbers (we specify that since $\mathcal{A}$ is not compact). On the other hand, by additivity of Euler characteristic for compact spaces, $\chi(\bar{A}) = 2\chi(\mathcal{A}) = 2 - 2l$ (recall that $\bar{A}$ is a compactification of A in a suitable toric surface, see Section 2). But $\chi(\bar{A}) = 2 - 2g$ and, therefore, $l = g$.

To ensure that $\mathbb{R}A$ has the right number of non-compact components we recall that $\bar{A}$ intersect the complexification of l_k in d_k points. Each such intersection corresponds to a "tentacle" of $\mathcal{A}$ which goes to infinity (see [3]). Therefore $\mathbb{R}^2 \setminus \mathcal{A}$ has $\sum\limits_{k=1}^{n} d_k$ non-compact components and each of them must be bounded by a non-compact component of $\mathbb{R}A$.

To finish the proof in the case when A is non-singular we need to show that these $g + \sum_{k=1}^{n} d_k$ components of $\mathbb{R}A$ are arranged in $(\mathbb{R}^*)^2$ in the Harnack way. This follows from Lemma 11 of [7]. Compactifying with $l_1 \cup \dots \cup l_n$ we obtain that $(\mathbb{R}T_\Delta; \mathbb{R}\bar{A}, l_1 \cup \dots \cup l_n)$ is a Harnack arrangement.

Now we consider a general case where A might have singular points.

Lemma 6. *A has no singularities other than real isolated double points.*

Proof. We claim that the singular points of A may only arise as the intersection points of two non-singular branches of $\tilde{A}$. Consider the map $\tilde{A} \to A \to \mathcal{A}$.

Over $\mathcal{A} \setminus F$ each of the two branches of $\tilde{A}$ must be non-singular. Indeed, it maps 1-1 to $\mathcal{A} \setminus F$ and, therefore, the link of each point of this branch is an unknot.

By a similar reason branches of $\tilde{A}$ cannot have singular points over F. Indeed, the links of such points are unknots since neighborhoods of those points map 2-1 to small half-disks from $\tilde{A}/\operatorname{conj}_{\tilde{A}}$.

By Lemma 1 of [7] the image of each branch of $\tilde{A}$ under Log has a convex complement. Therefore the images of branches of $\mathbb{R}\tilde{A}$ cannot intersect (that would produce points of $\mathcal{A}$ with at least 4 inverse images under $\operatorname{Log}|_A$).

Thus the only singularities of A are intersection points p of a pair of conjugate non-singular imaginary branches. If these branches are not transverse then they have a real tangent line τ. The points of τ close to p will be covered at least twice by each of the two branches of $\tilde{A}$ which leads to a contradiction. We conclude that the only singularities of A are A_1^+-singularities. $\square$

Now we may replace each A_1^+-point with a small oval that corresponds to its local perturbation and proceed similar to the case of non-singular curves.

5.2 Implication $3 \implies 2$.

This implication is contained in the proof of the main theorem in [7]. Indeed, a Harnack curve is in cyclically maximal position (see Theorem 3 of [7]). By Lemmas 5 and 8 of [7] we know that $F = \operatorname{Log}(\mathbb{R}A) = \partial\mathcal{A}$ and by Lemma 9 $\operatorname{Log}|_{\mathbb{R}A}$ is an embedding. Therefore the only singularities of $\operatorname{Log}|_A$ are folds and $\operatorname{Log}|_A$ is at most 2-1.

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